RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College under University of Calcutta)

B.A./B.SC. SIXTH SEMESTER EXAMINATION, MAY-JUNE 2013

THIRD YEAR

MATHEMATICS (Honours)

Date : 24/5/2013

Time : 11 am – 3 pm

Paper : VII

Full Marks : 100

[Use separate Answer Books for each group]

Group – A

Answer **any five** questions : [5×10]

1. a) Is every continuous real valued function f defined on a closed and bounded interval [a,b], necessarily a function of bounded variation? Justify your answer. [3]

   b) Let $f:[a,b]\to\mathbb{R}$ be a function of bounded variation defined on a closed and bounded interval [a,b]. Define the variation function V of f on [a,b]. Prove that V is monotone increasing on [a,b]. [3]

   c) Let $f:[a,b]\to\mathbb{R}$. Prove that f is a function of bounded variation on [a,b] if and only if f can be expressed as the difference of two monotone increasing functions on [a,b]. [4]

2. a) Let $f:[a,b]\to\mathbb{R}$ be bounded on [a,b] and let f be continuous on [a,b] except for a finite number of points in (a,b). Show that f is Riemann integrable on [a,b]. [4]

   b) Let $f:[a,b]\to\mathbb{R}$ be Riemann integrable on [a,b]. Show that there exists $c\in(a,b)$ such that $\int_a^c f(t)dt=\int_c^b f(t)dt$. [3]

   c) Using second mean value theorem (Weierstrass form) of integral calculus, show that

   $$\left|\int_a^b \frac{\cos x}{1+x}dx\right|<\frac{4}{1+a} \text{ where } b>a>0$$ [3]

3. a) Let $f:[1,3]\to\mathbb{R}$ be defined as follows :

   f(x) = 1, if $1\le x<2$

   = 2, if $2\le x\le 3$

   State with reasons :

   i) Whether f is Riemann integrable on [1,3]

   ii) Whether the fundamental theorem of integral calculus is applicable to f in [1,3] [1+2]

   b) With suitable example accompanied by proper justification, show that a function may be Riemann integrable on a closed and bounded interval without having a primitive in the interval. [3]

   c) State Lebesgue's theorem on the necessary and sufficient condition of Riemann Integrability of a function.

   Using this theorem prove that the following function f is Riemann integrable on [0,1], where $f:[0,1]\to\mathbb{R}$ is defined by

   f(0) = 0

   $f(x) = (-1)^{r-1}, \frac{1}{r+1}<x\le\frac{1}{r}$ for r = 1, 2, 3, ... [2+2]

4. a) Let $f:[a,b]\to\mathbb{R}$ be continuous on a closed and bounded interval [a,b]. Let $F:[a,b]\to\mathbb{R}$ be defined by $F(x)=\int_a^x f(t)dt$. Prove that F is differentiable on [a,b] and $F'(x)=f(x)$ for all $x\in[a,b]$. [4]

b) Let $f:[a,b]\to\mathbb{R}$ be continuous on [a,b] and $\phi:[a,b]\to\mathbb{R}$ be an antiderivative of f on [a,b]. Prove that $\int_a^b f(x)dx = \phi(b)-\phi(a)$. [3]

c) Show that $\frac{\pi^2}{9} < \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}dx < \frac{2\pi^2}{9}$. [3]

5. a) If f and g be two positive valued functions in [a,b] such that both have infinite discontinuity at 'a' only, both are integrable on $[a+\in, b]$, $0<\in<b-a$ and $\lim_{x\to a+}\frac{f(x)}{g(x)}=\ell$, where ℓ is a non zero finite number. Prove that $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ converge on diverge together. [4]

b) Show that $\int_{-1}^{1}\frac{dx}{x^3}$ is not convergent but its Cauchy principal value exists. [3]

c) Using Dirichlet's test prove that $\int_0^\infty \frac{\sin x}{x}dx$ is convergent. [3]

6. a) Prove that $\int_0^1 \frac{dx}{(1-x^6)^{1/6}} = \frac{\pi}{3}$. [3]

b) State Dirichlet's conditions for the convergence of the Fourier Series corresponding to a function f defined on $[-\pi,\pi]$. [2]

c) Obtain the Fourier Series expansion of the function f(x) = xsinx on $[-\pi,\pi]$. Hence deduce that $\frac{1}{2}+\frac{1}{1\cdot3}-\frac{1}{3\cdot5}+\frac{1}{5\cdot7}-\ldots=\frac{\pi}{4}$ [5]

7. a) Change the order of integration and evaluate : $\int_0^\infty\left\{\int_{y=x}^\infty \frac{e^{-y}}{y}dy\right\}dx$. [5]

b) Prove that $\iiint_E z^2 dxdydz = \frac{2\pi a^5}{15}$ where E is the region of the hemisphere $z\ge 0,\ x^2+y^2+z^2\le a^2$. [5]

8. a) Compute the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. [4]

b) Answer **<u>either (i) or (ii)</u>**. [3]

i) State with proper justification whether the following statement is true or false :
The plane curve $\nu=(f,g)$, where $f,g:[0,1]\to\mathbb{R}$ are defined by

$$f(x) = x^2\sin\frac{1}{x};\ x\ne 0$$
$$= 0 \quad ;\ x=0$$ and g(x) = x+1

is rectifiable.

ii) Find the length of one arc of the cycloid x = a(t – sint), y = a(1 – cost).

c) Prove that the improper integral $\int_a^\infty f(x)dx$ converges if and only if to any $\in>0$ there corresponds a number X such that $\left|\int_{X_1}^{X_2} f(x)dx\right|<\in$ for all $X_1, X_2 > X$. [3]

Group – B

Answer **any three** questions from **Q.No 9-13** and **any four** questions from **Q.No 14-19** :

9. a) State and prove Bayes' theorem on conditional probabilities. [2+3]

b) A and B are to independent witness in a particular case. The probability that A will speak the truth is x and the probability that B will speak the truth is y. A and B agree in a particular statement. Show that the probability that the statement is true is $\frac{xy}{1-x-y+2xy}$. [5]

Or,

A problem is given to three students whose chances of solving it are $\frac{1}{2},\frac{1}{3}$ and $\frac{1}{4}$. What is the probability that exactly two of them can solve the problem? Also, find the probability that the problem will not be solved. [3+2]

10. a) Define the distribution function of a random variable X. [2]

b) Show that the distribution function is a monotonic non-decreasing function, continuous on the right at all points but it may have a Jump discontinuity on the left of the point. [4]

c) A random variable X can assume the values –1,0,1 with probability $\frac{1}{3},\frac{1}{2},\frac{1}{6}$ respectively. Determine the distribution function. [4]

11. a) Find the mean and variance of the Binomial (n,p) distribution. [6]

b) Find the characteristic function of the gamma distribution. [4]

12. a) If the random variables X_1, X_2, ...X_n are mutually independent Gamma variates with parameters $\ell_1,\ell_2,...\ell_n$ respectively, then show that their sum $S_n = X_1 + X_2 + ... + X_n$ is also a Gamma variate with parameter $S_n = X_1 + X_2 + ... + X_n$, where $\ell = \ell_1 + \ell_2 + ... + \ell_n$. [5]

b) What that the acute angle θ between the regression lines is given by $\tan\theta = \frac{1-\rho^2}{|\rho|}.\frac{\delta_x\delta_y}{\delta_x^2+\delta_y^2}$.

Discuss the cases when $\rho = 0$, $\rho = \pm 1$ (the symbols are of usual meaning). [3+1+1]

Or,

Using Tchebycheff's in equality, show that in 1000 throws of an unbiased coin the probability that the number of heads lies between 450 and 550 is at least $\frac{9}{10}$. [5]

13. a) State the axioms of probability. [3]

b) Define conditional probability P(A/B) for any two events A and B connected to a random experiment E and P(B) > 0. [3]

c) Show that conditional probability satisfies all the axioms of probability. [4]

14. a) If f(z) is analytic at z_0, prove that it must be continuous at z_0.

b) Is the converse always true? Justify your answer. [3+2]

15. Show that the following function f is not differentiable at the origin even though it satisfies Cauchy-Riemann equation there :

$$f(z) = \frac{xy^2(x+iy)}{x^2+y^4} \quad ; \quad z \neq 0$$
$$= 0 \quad ; \quad z = 0$$
[5]

16. a) Let f(z) = u(x,y) + iv(x,y) and let $\lim_{h\to 0} \text{Re}\left\{\frac{f(z_0+h)-f(z_0)}{h}\right\}$ exist where $z_0 = x_0 + iy_0$; $x_0, y_0 \in \mathbb{R}$.

Show that $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ both exist at (x_0,y_0) and are equal. [3]

b) Given z = 3+2i a point in the complex plane, find the corresponding point on the Riemann sphere to be specified by you. [2]

17. If $G(\subset \mathbb{C})$ is a region and $f : G \to \mathbb{C}$ is analytic in G with $f'(z) = 0$, prove without using Cauchy-Riemann equations that f(z) is a constant. [5]

18. Expand f(z) = sinz in a power series in z, and also determine the region of convergence of this series. [4+1]

19. Prove that both the power series $\sum_{n=0}^{\infty} a_n z^n$ and its first time derived power series $\sum_{n=1}^{\infty} n a_n z^{n-1}$ have the same radius of convergence. [5]